Bound Tightening

It is not generally worthwhile to reduce the number of constraints as sometimes they might be necessary for a strong description of the feasible region. Instead, it is better to tighten the bounds (تضيق الحدود) if possible.

By tightening bounds the corresponding LP relaxation may be made more constrained and resulting in the optimal solution to the LP problem being closer to the optimal IP solution.

Comments

• There is no unique way of conducting bound tightening procedures, as the sequence in which the bound tightening is carried out has great impact to the result. لا

• Bound tightening procedure is not cheap to conduct especially when nonlinear constraints are present.

• Sometimes we can directly obtain optimal integer solution through bound tightening.

Procedure: Bound Tightening

For EVERY equality/inequality constraint $\sum_{i\in N} a_i x_i \leq b_i$:

Step 0: Initialize the variable counter k ← 0.

Step 1: Divide the constraint by the coefficient of variable x_k (if $a_k \neq 0$):

$$x_k + \sum_{i\neq k} \frac{a_i}{a_k} x_i \leq \quad b_i/a_k, \quad \text{if } a_k > 0$$

$$x_k + \sum_{i\neq k} \frac{a_i}{a_k} x_i \geq \quad b_i/a_k, \quad \text{if } a_k < 0$$

Step 2: Assume the upper and lower bound for x_i are denoted by U_i and L_i. Replace each of x_i other than x_k by U_i/L_i accordingly to the sign of its coefficient:

$$x_k + \sum_{i\neq k, \frac{a_i}{a_k}\geq 0} \frac{a_i}{a_k} L_i + \sum_{i\neq k, \frac{a_i}{a_k}\leq 0} \frac{a_i}{a_k} U_i \leq \quad b_i/a_k,$$

$$x_k + \sum_{i\neq k, \frac{a_i}{a_k}\leq 0} \frac{a_i}{a_k} L_i + \sum_{i\neq k, \frac{a_i}{a_k}\geq 0} \frac{a_i}{a_k} U_i \geq \quad b_i/a_k,$$

Step 3: Move everything to the right and round the r.h.s. up/down to the nearest integer value:

$$x_k \leq \lfloor b_i/a_k - \sum_{i\neq k, \frac{a_i}{a_k}\geq 0} \frac{a_i}{a_k} L_i - \sum_{i\neq k, \frac{a_i}{a_k}\leq 0} \frac{a_i}{a_k} U_i \rfloor,$$

$$x_k \geq \lceil b_i/a_k - \sum_{i\neq k, \frac{a_i}{a_k}\leq 0} \frac{a_i}{a_k} L_i - \sum_{i\neq k, \frac{a_i}{a_k}\geq 0} \frac{a_i}{a_k} U_i \rceil,$$

If $k < n$, advance $k \leftarrow k + 1$ and return to Step 1; otherwise continue to Step 4.

Step 4: If there is a non-trivial (that does not equal 1) greatest common divisor amongst ALL nonzero $|a_i|$, denote it by V_i ($V_i > 0$):

$$\sum_{i\in N} \frac{a_i}{V_i} x_i \leq \frac{b_i}{V_i}$$

Therefore

$$\sum_{i\in N} \frac{a_i}{V_i} x_i \leq \left\lfloor \frac{b_i}{V_i} \right\rfloor$$

Example:
Consider the constraint $4x_1 + 6x_2 \leq 9$, with $x_1, x_2 \geq 0$ and integer.
Analytically we can re-write the inequality constraint in terms of an expression of each variable: we choose x_1 so we divide the inequality by 4

$$x_1 \leq \tfrac{1}{4}(9 - 6x_2) = \tfrac{9}{4} - \tfrac{6}{4}x_2 \leq \left\lfloor \frac{9}{4} \right\rfloor = 2$$

As x_1 takes only positive integer values ($0 \leq a_i/a_k$ we take L_i), we can reduce the r.h.s. of above inequality to its integer value to make $x_1 \leq 2$.
Similarly, as we choose x_2 so we divide the inequality 6

$$x_2 \leq \tfrac{1}{6}(9 - 4x_1) = \tfrac{9}{6} - \tfrac{6}{6}x_1 \leq \left\lfloor \frac{6}{4} \right\rfloor = 1$$

We have $x_2 \leq 1$. gcd = max{1, 2}=2.

Moreover, observe that the greatest common divisor of all variable coefficients is 2, we can divide both sides of the constraint by it to obtain a tighter constraint:

$$\frac{4}{2}x_1 + \frac{6}{2}x_2 \leq \frac{9}{2},$$

Then we obtain $2x_1 + 3x_2 \leq 4$.

Exercise: Find bounds tightening for the IP problem.

$$\begin{aligned}
\min \quad & 5x_1 + 7x_2 + 10x_3 + 3x_4 + x_5 \\
\text{s.t.} \quad & x_1 - 3x_2 + 5x_3 + x_4 - x_5 \geq 2, \\
& -2x_1 + 6x_2 - 3x_3 - 2x_4 + 2x_5 \geq 0, \\
& -x_2 + 2x_3 - x_4 - x_5 \geq 1, \\
& x \in \{0,1\}^5
\end{aligned}$$

Remarks:
This example is an extreme case of the effect of bound tightening in an IP since this procedure alone completely solves the problem.

2.3 Special-purpose Algorithm for Knapsack Model

The structure invoked in this section is that the problems have only one constraint other than bounds and integrality on variables. A typical application is the **Knapsack model.**

$$\begin{aligned}
\max \quad & \sum_{j \in N} v_j x_j \\
s.t. \quad & \sum_{j \in N} c_j x_j \leq b, \qquad \text{(capacity conditions)} \\
& x \in \{0,1\}^n \quad (or\ x \in Z_+^n)
\end{aligned}$$

As Knapsack been known as the simplest model in IP, in this section we will talk about two solution methods of it.

Simple Greedy Heuristic

Heuristic (or approximate algorithms) are designed to find good, but not necessarily optimal, solutions quickly.

In Knapsack problem we are aiming to pack items to maximize total values subject to a single capacity constraint. Intuitively, one may start from selecting the ”smallest” item with the ”highest” possible value.
This leads to the idea of considering items in order of their value-volume ratio

$$\frac{v_j}{c_j}, \forall j \in N$$

Without loss of generality, suppose that throughout this section we always have:

$$\frac{v_1}{c_1} \geq \frac{v_2}{c_2} \geq \cdots \geq \frac{v_n}{c_n}$$

- For **integer Knapsack problem**, its optimal solution to LP relaxation is

$$x_1 = \frac{b}{c_1}, x_2 = \cdots = x_n, with\ z_{LP}^* = \frac{v_1 b}{c_1}$$

Which (immediately) implies

$$\bar{x}_1 = \left\lfloor \frac{b}{c_1} \right\rfloor, \bar{x}_2 = \cdots = \bar{x}_n, with\ \ \bar{z}_{IP} = v_1 \left\lfloor \frac{b}{c_1} \right\rfloor$$

By Principle 2.3 and 2.6, we have

$$v_1 \left\lfloor \frac{b}{c_1} \right\rfloor = \bar{z}_{IP} \leq z_{IP}^* \leq z_{LP}^* \ = \frac{v_1 b}{c_1}$$

Furthermore for the remainder (if there is) of capacity $\tilde{b} = b - \ c_1 \left\lfloor \frac{b}{c_1} \right\rfloor$, we can re-formulate the Knapsack problem as

$$\max \quad \sum_{j \geq 2, c_j \leq \tilde{b}} v_j x_j$$
$$s.t. \quad \sum_{j \geq 2, c_j \leq \tilde{b}} c_j x_j \ \leq \tilde{b}, \quad x_j \in \{0, 1\}$$

Continue the algorithm until no more items can be added.

- For 0-1 **Knapsack problem**, its optimal solution to LP relaxation is

$$x_i = \begin{cases} 1, & if\ i \le k; \\ \frac{b - \sum_{j=1}^{k} c_j x_j}{c_{k+1}}, & if\ i = k+1; \\ 0, & otherwise \end{cases} \qquad i = 1, 2, \dots, n$$

Where k is the largest integer where $\sum_{j=1}^{k} c_j x_j \le b$, which implies

$$\bar{x}_1 = \cdots = \bar{x}_k = 1, \bar{x}_{k+1} = \cdots = \bar{x}_n = 0\ with\ \bar{z}_{IP} = \sum_{j=1}^{k} v_j x_j$$

and bounds

$$\sum_{j=1}^{k} v_j x_j = \bar{z}_{IP} \le z_{IP}^* \le z_{LP}^* = \sum_{j=1}^{k} v_j x_j + \frac{v_{k+1}}{c_{k+1}} (b - \sum_{j=1}^{k} c_j x_j)$$

Then we can re-formulate the model in a similar fashion as the integer Knapsack.

We can NOT guarantee the solution found by this greedy heuristic is optimal, although from the upper/lower bound derived one may observe small sub-optimality.

Exercise: Consider the following 0-1 Knapsack problem

$$\max \quad 12\,x_1 + 16x_2 + 9x_3 + 10x_4 + 8x_5$$

$$\text{s.t} \quad 3\,x_1 + 3x_2 + 2x_3 + 2x_4 + 2x_5 \le 6,$$

$$x \in \{0, 1\}^5$$

1. Solve it approximately by **Greedy Heuristic**, derive sub-optimality gap on your final solution.

   Solution

   Rearranging items in descending order of their $\frac{v_j}{c_j}$ rates:

$$\frac{v_2}{c_2} \ge \frac{v_4}{c_4} \ge \frac{v_3}{c_3} \ge \frac{v_1}{c_1} = \frac{v_5}{c_5}$$

$$5.3 \ge 5 \ge 4.5 \ge 4 \ge 4$$

- Choose 5.3 then $x_2 = 1, and\ \tilde{b} = b - c_2 = 6 - 3 = 3$ (we take away the item weight from the back weight)
- Choose 5 then $x_4 = 1, and\ 3 - 2 = 1$

Now $\tilde{b} = b - c_2 - c_4 = 1$

- Choose 4.5 then $x_3 = \frac{b - \sum_{j=1}^{k} c_j x_j}{c_{k+1}} = \frac{6-3*1-2*1}{2} = \frac{1}{2}$, because the remainder weight in the back that we can fill is 1 kg so we can't take all the weight of $c_3 = 2$, we can fill only 1 kg. So we take $x_3 = \frac{1}{2}$. So $\tilde{b} = b - c_2 - c_4 - c_3 = 6 - 3 - 2 - 1 = 0.$

As there is no item in the remaining set (item 1, 5), we can't add more items to the back so the rest of the variables $x_1,\ x_5$ will take 0.

- We check the constraint.
- $3 \times 0 + 3 \times 1 + 2 \times \frac{1}{2} + 2 \times 1 + 2 \times 0 \leq 6$
- The objective function is
- $z_{LP}^* = 12 \times 0 + 16 \times 1 + 9 \times \frac{1}{2} + 10 \times 1 + 8 \times 0 = 30.5$

Or by $z_{LP}^* = \sum_{j=1}^{k} v_j x_j + \frac{v_{k+1}}{c_{k+1}}\left(b - \sum_{j=1}^{k} c_j x_j\right) = \left(16 * 1 + 10 * 1 + 9 * \frac{1}{2}\right) + \frac{12}{3} * \left(6 - (3 + 2 + 1)\right) = 30.5$

In first iteration we have the optimal solution to LP relaxation is

$$x_2 = x_4 = 1, x_3 = \frac{1}{2}, x_1 = x_5 = 0, with\ z_{LP}^* = 30.5$$

Round off we get:

$$x_2 = x_4 = 1, x_1 = x_3 = x_5 = 0, with\ z_{IP} = 26$$

Having a smaller weight than $\tilde{b}$, we stop and report the solution:

$$x_2 = x_4 = 1, x_1 = \ x_3 = x_5 = 0, with\ 26 \leq \ z_{IP}^* \ \leq 30$$

(Note that this is not the optimal solution!)

Disaggregating[1] Constraints فصل القيود (تفكيك)

Disaggregation works on pure 0-1 constraints, i.e. constraints which involve only 0-1 variables. In this section we consider the **0-1 Knapsack** model and try to disaggregate its single constraint:

$$\sum_{j \in N} c_j x_j \leq b.$$

Definition 2.8 A subset $\{i_1, i_2, \dots, i_r\}$ with r elements taken from indices $\{1, 2, \dots, n\}$ is a cover for 0-1 Knapsack problem if $c_{i_1} + c_{i_2} + \dots + c_{i_r} > b.$

Definition 2.9 A **cover** is a **minimal cover** if no proper subset of $\{i_1, i_2, \dots, i_r\}$ also a cover.

It is obvious that not all of the x_i is that are included in a cover can be equal to 1 *simultaneously. So we can write the constraint:*

$$x_{i_1} + x_{i_2} + \dots + x_{i_r} \leq r - 1$$

Procedure: Minimal Cover Extension for set $\{i_1, i_2, \dots, i_r\}$

Step 1: Let c_{max} be the largest coefficient c_{i_j} where i_j is a member of $\{i_1, i_2, \dots, i_r\}$.

Step 2: Consider each of the elements that were not in set $\{i_1, i_2, \dots, i_r\}$. Find all of those elements $k_1, \dots, k_l$ such that $c_{k_j} \geq c_{max}$, $j = 1, \dots, l$.

Step 3: Append $k_1, \dots, k_l$ to $\{i_1, i_2, \dots, i_r\}$ to produce the extended cover.

Definition 2.10 If a minimal cover gives rise to an extended cover which is not a proper subset of any other extended cover arising from a minimal cover with the same number of indices, the original minimal cover is known as a **strong cover**.

Constraint derived from the **strong cover** is then:

$$x_{i_1} + \dots + x_{i_r} + x_{i_{r+1}} + \dots + x_{i_{r+s}} \leq r - 1$$

[1] separate (something) into its component parts

2. **Disaggregating constraints** to get a tighter description of its feasible region.

Solution:
For the constraint $3x_1 + 3x_2 + 2x_3 + 2x_4 + 2x_5 \leq 6$

Consider r = 1, 2, there is no cover exists.

r = 3, **Minimal Covers** are:
{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}, {1, 3, 5}, {1, 4, 5}, {2, 3, 4}, {2, 3, 5}, {2, 4, 5}.
(Note that {3, 4, 5} is not a cover as $c_3 + c_4 + c_5 = 6 = r.h.s.$)

Extend covers:
{1, 2, 3} $\longrightarrow$ {1, 2, 3}, {1, 2, 4} $\longrightarrow$ {1, 2, 4}
{1, 2, 5} $\longrightarrow$ {1, 2, 5}, {1, 3, 4} $\longrightarrow$ {1, 2, 3, 4}
{1, 3, 5} $\longrightarrow$ {1, 2, 3, 5}, {1, 4, 5} $\longrightarrow$ {1, 2, 4, 5}
{2, 3, 4} $\longrightarrow$ {1, 2, 3, 4}, {2, 3, 5} $\longrightarrow$ {1, 2, 3, 5}
{2, 4, 5} $\longrightarrow$ {1, 2, 4, 5}

Strong covers:
{1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 4, 5}
Which yield the constraints where $r - 1 = 2$

$$x_1 + x_2 + x_3 + x_4 \leq 2$$

$$x_1 + x_2 + x_3 + x_5 \leq 2$$

$$x_1 + x_2 + x_4 + x_5 \leq 2$$

H.W: Consider the following 0-1 Knapsack problem

$$\max \quad x_1 + 2x_2 + 5x_3 + 6x_4$$

$$\text{s.t} \quad 2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 8,$$

$$x \in \{0,1\}^4$$

1-Solve it approximately by **Greedy Heuristic**.

2-Disaggregating constraints to get a tighter description of its feasible region.